Supplementary Note for *Walrasian Equilibrium in Large, Quasi-linear Markets*

Eduardo M. Azevedo* E. Glen Weyl† Alexander White‡

February 28, 2013

This note shows that, within the quasi-linear framework we consider, economies with a large but finite number of consumers admit prices that clear the market approximately.

1 Economies with Finite Numbers of Consumers

Finite economies are identical to the continuum economies in the main text regarding preferences and goods, but have a finite collection of utilities $\mathscr{U} = \{u_1, \ldots, u_U\}$; each utility has a finite number of agents, n_i assigned to it. Thus the total (finite) number of agents in the economy $N = \sum_{i=1}^{U} n_i$.

The endowment vector is a vector of rational numbers in $(0, 1)$ with denominators that are integral factors of N. The interpretation is that an economy with endowment q has a vector of units of goods available corresponding to Nq. An allocation is a map $\mathbf{x} : \mathscr{U} \to \Delta X$ with the property that $\mathbf{x}(u_i)$ has denominators in all of its coordinates that are integral factors of n_i. Define the G-dimensional vector of the measure of each good consumed $\tilde{\mathbf{x}}$ as in the main text. An allocation is feasible if

$$\sum_i n_i \tilde{\mathbf{x}}(u_i) = Nq.$$

A *m-replication* of an economy, for a positive integer N, is the economy where each n_i is multiplied by m. Demand sets of an m-replica $D^m(p, u_i)$ are defined as before, ex-

*The Wharton School at the University of Pennsylvania; eazevedo@wharton.upenn.edu

†University of Chicago Department of Economics & Toulouse School of Economics; weyl@uchicago.edu

‡Tsinghua University School of Economics and Management; awhite@sem.tsinghua.edu.cn

cept that, again, only rational allocations with appropriate denominators are allowed. A competitive equilibrium is a price-allocation pair $(p, \mathbf{x})$ such that $\mathbf{x}$ is feasible, and for each i we have $\mathbf{x}(u_i) \in D(p, u_i)$. The *continuum replication* of a finite economy is the continuum economy defined by $\eta(S) = \sum_i \frac{n_i}{N} \delta_{u_i}$ and the endowment vector q; for clarity we denote the demand set in the continuum replication, the convex hull of $D^1(p, u_i)$, as $D^\infty(p, u_i)$.

Aggregate demand of the m replica is defined as

$$D^m(p) = \{\sum_i \frac{n_i}{N} \cdot \tilde{\mathbf{x}} : \mathbf{x}(\mathbf{u_i}) \in D^m(p, u_i) \text{ for all } i\}.$$

Aggregate demand of the continuum replication is defined as in the main text and denoted D^∞.

The following Proposition summarizes the relationship between continuum and discrete economies.

Proposition 1. *Any equilibrium price of an m-replica is an equilibrium price of the ∞ replica (this is true in particular for $m = 1$).*

Any equilibrium price p^* of the ∞ replica approximatelly clears the market in finite but large replicas. That is, there exist a sequence of $d^m \in D^m(p^*)$ such that

$$||d^m - q||_\infty \to 0$$

as m converges to infinity.

Proof. First note that an equilibrium price of the m-replica is simply a price vector such that $q \in D^m(p)$. Note moreover that, by definition, $D^\infty(p) \supseteq D^m(p)$. That is, because in the infinite replica agents of a given type can demand arbitrary mixtures over bundles, while the mixtures are restricted in m-replicas by divisibility constraints, the ∞-replica demand correspondence has a strictly larger image. This observation proves that every equilibrium price of the m-replica is an equilibrium price of the ∞-replica.

For the second part, note that $D^\infty(p^*)$ is a convex set, generated by convex combinations of a finite set of extreme points. Moreover, $q \in D^\infty(p^*)$, as p^* is an equilibrium for the ∞-replica. The set $D^m(p^*)$ is composed of a finite set of points, and is a subset of the polytope $D^\infty(p^*)$. Note, however, that $D^m(p^*)$ must contain all convex combinations of the extreme points of $D^\infty(p^*)$ such that the weights are multiples of

$1/m$. Therefore, the distance between q and $D^m(p^*)$ converges to 0 as m grows. $\square$

We end this section with a curious observation. Consider an equilibrium p^* of the continuum replica. It is also true that, for infinitely many values of m, p^* is an exact equilibrium of the m-replica. To see this, note that since q is in the convex set $D^\infty(p^*)$, it can be written as a convex combination of the extreme points of $D^\infty(p^*)$. That is,

$$q = \sum_i \alpha_i d_i,$$

where the $\alpha_i \in (0,1)$, $\sum_i \alpha_i = 1$, and the d_i are extreme points of $D^\infty(p^*)$. Without loss of generality, we may take the d_i to be linearly independent. Let d be the matrix whose columns are the d_i, and α the column vector of coordinates α_i. We then have

$$\begin{aligned} q &= d\alpha, \text{ and consequently} \\ d'q &= d'd\alpha. \end{aligned}$$

Since the d_i are linearly independent, the matrix $d'd$ is left-invertible. Therefore,

$$\alpha = (d'd)^{-1}d'q.$$

In particular, this implies that the α_i are rational numbers. Therefore, there exists m_0 such that all α_i are integer multiples of $1/m_0$. Since $D^m(p^*)$ includes all convex combinations of the vertexes of $D^\infty(p^*)$ with weights multiple of $1/m$, then $D^m(p^*)$ will include q for all m that are multiples of m_0. In the example in the text, we had found that exact equilibria existed for every even replication of the economy. The reasoning above shows that this is a more general phenomenon.